MAT141 - Mean Value Theorem

Objectives:
1. To become familiar with the Mean Value Theorem (MVT).
2. To understand the conditions and meaning of the MVT.
3. To use graph.cgi and maxima to find points that satisfy the MVT in a given interval.

Background:
The Mean Value Theorem states: let $f$ be a function such that:
(i) it is continuous on the interval $[a, b]$
(ii) it is differentiable on the open interval $(a, b)$
Then there is a number $c$ in the open interval $(a, b)$ such that: $f'(c) = \frac{f(b) - f(a)}{b - a}$

Interpretation of the MVT: The points $(a, f(a))$ and $(b, f(b))$ are two points on the graph of $f(x)$. It follows that if you connect these two points with a straight line, then this line has a slope equal to $\frac{f(b) - f(a)}{b - a}$. The MVT simply states that there must be a point somewhere between $a$ and $b$ on the graph of $f(x)$ where the slope of the tangent line equals the slope of the line connecting $a$ and $b$.

Lab: You will be verifying the MVT for a few functions in a few different intervals. For each function:

a. Plot the function using graph.cgi.
b. Decide upon any interval $[a, b]$ where $f(x)$ is continuous and differentiable.
c. Evaluate $f(a)$ and $f(b)$. This will give you two points on the graph of $f(x)$.
d. Compute the slope ($m$) of the line connecting the points $(a, f(a))$ and $(b, f(b))$.
e. Compute the equation of the line containing these two points and plot. Use $y - y_1 = m(x - x_1)$ and simplify to $y = mx + b$.
f. Compute the derivative of $f(x)$.
g. Set the derivative of $f$ equal to the slope found in part d. Solve this equation. You may use maxima and the "solve" or "realroots" functions to help you. Approximate all solutions good to two decimal points (you can use the "float" function in Maxima for this). List all solutions that fall in the open interval $(a, b)$. Select any one of these values and call it $c$.
h. Compute $f(c)$ good to two decimal places.
i. Compute the equation of the line containing $(c, f(c))$ using the slope computed in part d. Plot on the same set of axes.
j. Observe your graph. Is the line you plotted in part i the tangent to the graph of $f(x)$ at the point $(c, f(c))$? Is it parallel to the line connecting $(a, f(a))$ and $(b, f(b))$? If so, the MVT has been satisfied.
k. Print out your graph using an appropriately scaled window. Label all important lines and points on this graph. This is important documentation. Labs will not be graded if not appropriately labeled.

Use these functions:
1. $f(x) = x^2 + 4x$
2. $f(x) = \frac{(x+2)}{(x-1)}$
3. $f(x) = 5 \sin(x)$

Make sure you show all work on the same sheet as the graph. Make sure you use non-zero values for slope! Make sure the point of tangency lies between the two endpoints on the chosen interval. Good luck!!!
Sample Problem:

I chose the function \( f(x) = 3x^2 - 7 \) and an interval of \([-1, 2]\). Notice that the function is continuous and differentiable on this interval.

\( f(-1) = -4 \) and \( f(2) = 5 \). So the points on the graph are \((-1, -4)\) and \((2, 5)\). The slope of the line containing these points is:

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(5 + 4)}{(2 + 1)} = \frac{9}{3} = 3.
\]

So now I plot the line \( y - y_1 = m(x - x_1) \)
\( y + 4 = 3(x + 1) \) which is the same as \( y = 3x - 1 \)

At this point, I compute the derivative of \( f(x) \)
\( f'(x) = 6x \). I want to set \( f'(x) = \) the slope of the line computed above and solve. So:

\( 6x = 3 \), which means \( x = \frac{1}{2} \).
and \( f(1/2) = -6.25 \).

Compute the equation of the line containing \((1/2, -6.25)\) with a slope of 3 and plot.
\( y + 6.25 = 3(x - 1/2) \)
\( y = 3x - 7.75 \)

Label all points, lines and other important information. Good luck!